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G. Penso and V. Silvestrini: INVARIANT PHASE-SPACE INTEGRAL FOR A SYSTEM OF n-PARTICLES. -

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G. Penso[(x)] and V. Silvestrini: INVARIANT PHASE-SPACE INTEGRAL FOR A SYSTEM OF n-PARTICLES.

SUMMARY.

The phase-space integral for a system of n-particles is expressed in terms of the integral for n-1 particles. The formula is used for the explicit calculation of the phase-space integral for 3 and 4 particles. A formula is also given for the calculation of the effective-mass spectrum of k particles out of n.

1. DERIVATION OF THE ITERATIVE FORMULA.

The invariant[(+)] phase-space integral for a system of n-particles is given by:

(1) $$\phi_n(|P_0|) = \int \prod_{i=1}^{n} d^4 P_i \, \delta(P_0 - \sum_{i=1}^{n} P_i) \prod_{i=1}^{n} \delta(P_i^2 - m_i^2) \prod_{i=1}^{n} \theta(\epsilon_i)$$

where:

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(+) - Non-invariant phase space integrals are also used. See for istance M.M. Block, Phys. Rev. 101, 796 (1956).

$P_o \equiv (\vec{P}_o, E_o)$ is the total 4-momentum of the particles.

$(i=1,2...n)$
- $P_i \equiv (P_i, E_i)$ is the 4-momentum of the i^{th} particle
- m_i is the mass of the i^{th} particle
- $\theta(E_i)$ are defined as 0 for $E_i \leq 0$ and 1 for $E_i > 0$

$\delta(P_o - \sum_{i=1}^{n} P_i)$ accounts for 4-momentum conservation, $\delta(P_i^2 - m_i^2)$ gives the mass-shell condition for the i^{th} particle, and $\theta(E_i)$ gives the condition that E_i must be positive.

Using the property of the δ-function $\delta(a-b) = \int \delta(a-z)\delta(z-b)\,dz$, formula (1) can be put in the form:

(2)
$$\Phi_n(|P_o|) = \int d^4P^* d^4P_n \left[\int \prod_{i=1}^{n-1} d^4P_i \, \delta(P^* - \sum_{i=1}^{n-1} P_i) \prod_{i=1}^{n-1} \delta(P_i^2 - m_i^2) \prod_{i=1}^{n-1} \theta(E_i) \right] \cdot$$
$$\cdot \delta(P_o - P^* - P_n)\delta(P_n^2 - m_n^2)\theta(E_n) =$$
$$= \int d^4P^* d^4P_n \, \Phi_{n-1}(|P^*|)\delta(P_o - P^* - P_n)\delta(P_n^2 - m_n^2)\theta(E_n)$$

where $\phi_{n-1}(|P^*|)$ is the phase-space integral for a system of n-1 particles with 4-momentum P^*.

Remembering that

$$\delta(P_i^2 - m_i^2) = \frac{\delta(E_i + \sqrt{\vec{p}_i^{\,2} + m_i^2})}{2E_i} + \frac{\delta(E_i - \sqrt{\vec{p}_i^{\,2} + m_i^2})}{2E_i}$$

we integrate E_n and P^*. We get:

(3)
$$\xi_n(|P_o|, E_n, \theta_n, \phi_n)\, d^3\vec{P}_n = d^3\vec{P}_n \, \frac{\Phi_{n-1}(|P_o - P_n|)}{2E_n}$$

where now

$$E_n = \sqrt{\vec{p}_n^{\,2} + m_n^2}$$

(3) can be put in the form:

(4)
$$\xi_n \, dE_n \, d\cos\theta_n \, d\phi_n = \frac{|\vec{p}_n|}{2} \phi_{n-1}(|P_o - P_n|)\, d\cos\theta_n \, d\phi_n \, dE_n$$

ξ_n is the energy and angular distribution of the n^{th} particle. (4) can be easily integrated in the center of mass system of the n particles, to obtain the phase-space integral ϕ_n:

(5) $$\phi_n(|P_0|) = 2\pi \int_{E_{n\,min}}^{E_{n\,max}} |\vec{P}_n|\, dE_n\, \phi_{n-1}(|P_0 - P_n|)$$

with $E_{nmin} = m_n$; $\quad E_{nmax} = \dfrac{E_0^2 + m_n^2 - \left(\sum_{i=1}^{n-1} m_i\right)^2}{2E_0}$

Formulae (4) and (5) hold obviously only for $n > 2$. It is important to note that they are invariant, and that (5) is a function of $|P_0|$ only. They allow to calculate the phase-space integral for n-particles once it is known for n-1 particles. We will use them to calculate explicitly ξ_n and ϕ_n for n = 3,4.

2. PHASE-SPACE INTEGRAL FOR 2 AND 3 PARTICLES.

The phase-space integral for the case n=2 is easily obtained by direct integration of (1):

(6) $$\phi_2(|P_0|) = \frac{\pi}{2|P_0|^2}\left\{\left[|P_0|^2 - (m_1+m_2)^2\right]\left[|P_0|^2 - (m_1-m_2)^2\right]\right\}^{1/2}$$

By use of (4), (5) and (6) one gets:

(7) $$\xi_3(|P_0|, E_3, \theta_3, \phi_3)\, dE_3\, d\cos\theta_3\, d\phi_3 =$$
$$= \frac{\pi\, |\vec{P}_3|}{4|P_0-P_3|^2}\left\{\left[|P_0-P_3|^2 - (m_1+m_2)^2\right]\left[|P_0-P_3|^2 - (m_1-m_2)^2\right]\right\}^{1/2} dE_3\, d\cos\theta_3\, d\phi_3$$

(8) $$\phi_3(|P_0|) = \frac{\pi}{4}\int \frac{|\vec{P}_3|}{|P_0-P_3|^2}\left\{\left[|P_0-P_3|^2 - (m_1+m_2)^2\right]\left[|P_0-P_3|^3 - (m_1-m_2)^2\right]\right\}^{1/2} dE_3\, d\cos\theta_3\, d\phi_3$$

This integral can be more easily evaluated in the center of mass system of the 3-particles. In this case the function to be integrated is independent of θ_3 and ϕ_3, and we have:

(9) $$\phi_3(|P_0|) = \pi^2 \int_{E_{3\,min}}^{E_{3\,max}} \frac{|\vec{P}_3|\, dE_3}{|P_0-P_3|^2}\left\{\left[|P_0-P_3|^2 - (m_1+m_3)^2\right]\cdot \left[|P_0-P_3|^2 - (m_1-m_2)^2\right]\right\}^{1/2}$$

with:

$$\left.\begin{aligned} |P_o-P_3|^2 &= (E_o-E_3)^2-\vec{P}_3^2 \\ E_{3min} &= m_3 \\ E_{3max} &= \left[E_o^2+m_3^2-(m_1+m_2)^2\right]/2E_o \end{aligned}\right\} E_o = |P_o|$$

The solution of (9) involves complete elliptic integrals, but can easily be done numerically.

3. PHASE-SPACE INTEGRAL FOR 4 PARTICLES.

In the case of 4 particles, we have:

$$\xi_4(|P_o|,E_4,\theta_4,\varphi_4)\,dE_4\,d\cos\theta_4\,d\varphi_4 = \frac{|\vec{p}_4|}{2}\phi_3(|P_o-P_4|)\,dE_4\,d\cos\theta_4\,d\varphi_4 \tag{10}$$

$$\phi_4(|P_o|) = \int \frac{|\vec{p}_4|}{2}\phi_3(|P_o-P_4|)\,dE_4\,d\cos\theta_4\,d\varphi_4 \tag{11}$$

With ϕ_3 given by (9).

This integral can again be evaluated in the center of mass of the 4-particles; in this system (10) and (11) have the explicit form:

$$\begin{aligned} &\xi_4(|P_o|,E_4,\theta_4,\varphi_4)\,dE_4\,d\cos\theta_4\,d\varphi_4 = \\ &= \frac{\pi^2}{2}|\vec{p}_4|\,dE_4\,d\cos\theta_4\,d\varphi_4\int_{E_{3min}}^{E_{3max}}\frac{|\vec{P}_3|\,dE_3}{W^2}\left\{\left[W^2-(m_1+m_2)^2\right]\left[W^2-(m_1-m_2)^2\right]\right\}^{1/2} \end{aligned} \tag{12}$$

$$\begin{aligned} \phi_4(|P_o|) = 2\pi^3\int_{E_{4min}}^{E_{4max}}|\vec{P}_4|\,dE_4\int_{E_{3min}}^{E_{3max}}\frac{|\vec{P}_3|\,dE_3}{W^2}\,\cdot \\ \cdot\left\{\left[W^2-(m_1+m_2)^2\right]\left[W^2-(m_1-m_2)^2\right]\right\}^{1/2} \end{aligned} \tag{13}$$

where:

$$W^2 = \left(\sqrt{(E_o-E_4)^2-\vec{P}_4^2}-E_3\right)^2-\vec{P}_3^2$$

$$E_{3min} = m_3$$

$$E_{3max} = \frac{(E_0-E_4)^2 - \vec{p}_4^{\,2} + m_3^2 - (m_1+m_2)^2}{2\sqrt{(E_0-E_4)^2-\vec{p}_4^{\,2}}}$$

$$E_{4min} = m_4$$

$$E_{4max} = \frac{E_0^2 + m_4^2 - (m_1+m_2+m_3)^2}{2E_0}$$

As an example, ξ_4 has been calculated in a particular case, and it is shown in fig.1.

4. EFFECTIVE-MASS DISTRIBUTION.

Another problem of interest is the calculation of the effective-mass distribution, namely the following: given a reaction with n-particles in the final state, the calculation of the distribution of the effective mass, m^*, of k among the n particles ($m^* \equiv$ modulus of the total 4-momentum of the k particles).

In the case k=n-1, the problem is easily solved. In fact, using (4) and since:

$$|P_0-P_n| = m^*$$

$$dE_n = (m^*/E_0)dm^*$$

$$\left.\begin{aligned} \cos\theta_n &= -\cos\theta^* \\ \phi_n &= 2\pi-\phi^* \end{aligned}\right\} \quad \begin{aligned} &\theta^*, \phi^* \text{ angles of the total} \\ &\text{3-momentum of the n-1 particles} \end{aligned}$$

we get:

$$(14) \qquad \rho_{n,n-1}(m^*)\,dm^*\,d\cos\theta^*\,d\phi^* = |\vec{p}_n(m^*)|\,\frac{m^*}{E_0}\,\phi_{n-1}(m^*)\,dm^*\,d\cos\theta^*\,d\phi^*$$

In the case $k \geq 2$, $n-k \geq 2$, using a procedure similar to the one used to prove (4), the following relation can be proved:

$$(15) \qquad \phi_n(|P_0|) = \int d^4P^*\,\phi_k(|P^*|)\,\phi_{n-k}(|P_0-P^*|)$$

where P^* is the total 4-momentum of the k particles.

We have:

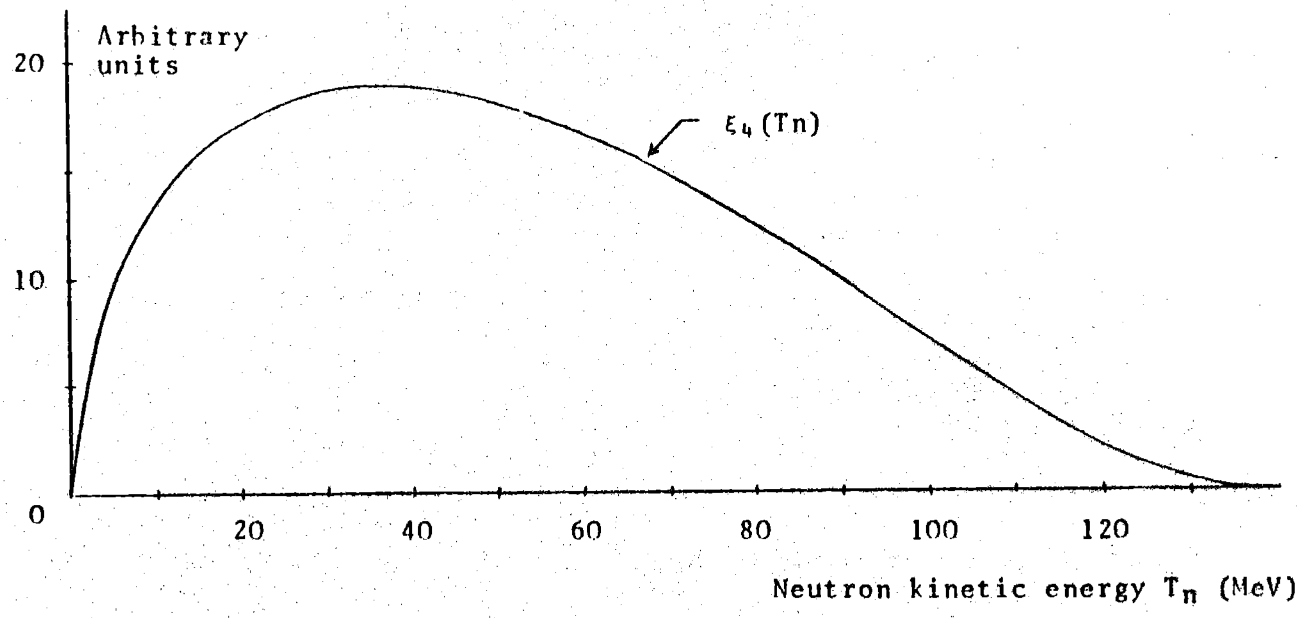

FIG.1. Energy **distribution of the neutron from the reaction**

$$\pi^- + p \rightarrow n + \pi^\circ + \pi^\circ + \pi^\circ ,$$

according to the invariant statistical theory. The distribution is calculated in the center-of-mass system, for a kinetic energy of the incident pion $T_{\pi^-} = 1.1$ Bev.

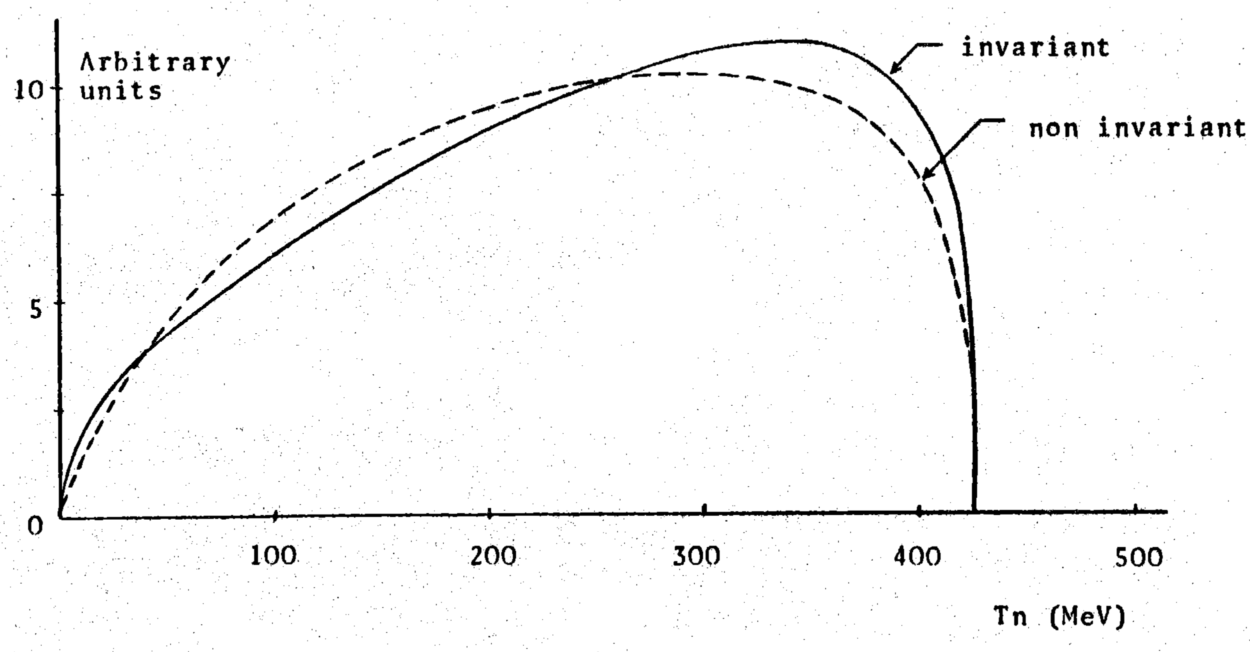

FIG.2. Energy distribution of the neutron from the reaction

$$\pi^- + p \rightarrow n + \pi^\circ + \pi^\circ .$$

This distribution is calculated in the lab. system, at an angle of 40°, for a kinetic energy of the π^- T_{π^-} = = 1.1 Bev. The invariant (full-line) and non-invariant[1÷4] (dotted) statistical theories have been used for comparison.

$$|P^*| \equiv m^*$$

$$\begin{aligned} d^4p^* = d^3\vec{p}^{\,*} dE^* &= \vec{p}^{\,*2} d|\vec{p}^{\,*}| d\cos\theta^* d\phi^* dE^* = \\ &= \vec{p}^{\,*2} \frac{d|\vec{p}^{\,*}|}{dm^*} dm^* d\cos\theta^* d\phi^* dE^* = \\ &= (E^{*2}-m^{*2})^{1/2} m^* dm^* d\cos\theta^* d\phi^* dE^* \end{aligned}$$

so that (15) becomes:

(16) $$\phi_n(|P_0|) = \int dm^* d\cos\theta^* d\phi^* dE^* \, m^* (E^{*2}-m^{*2})^{1/2} \phi_k(m^*) \phi_{n-k}(|P_0-P^*|)$$

If one integrates over dE^*, $d\cos\theta^*$, $d\phi^*$, but not over dm^*, one gets the effective-mass distribution for the k particles. These integrations are more easily evaluated in the center of mass system of the n-particles.

We get:

(17) $$\rho_{n,k}(m^*)dm^* = 4\pi m^* \phi_k(m^*) dm^* \int_{E^*_{min}}^{E^*_{max}} dE^* (E^{*2}-m^{*2})^{1/2} \phi_{n-k}(|P_0-P^*|)$$

where:

$$E^*_{min} = m^*$$

$$E^*_{max} = \left[E_0^2 + m^{*2} - \left(\sum_{k+1}^{n} m_i\right)^2\right] / 2E_0$$

The range of variation of m^* is between

$$m^*_{min} \quad \text{and} \quad m^*_{max},$$

with;

$$m^*_{min} = \sum_1^k m_i; \qquad m^*_{max} = E_0 - \sum_{k+1}^{n} m_i$$

For istance, in the case n=5, k=3, the problem can be easily solved using formulae (6) and (9): in fact in this case only ϕ_n with n=2 and n=3 appear in (17).

Thanks are due to Drs. G. Da Prato and G. De Franceschi for helpful discussions.

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